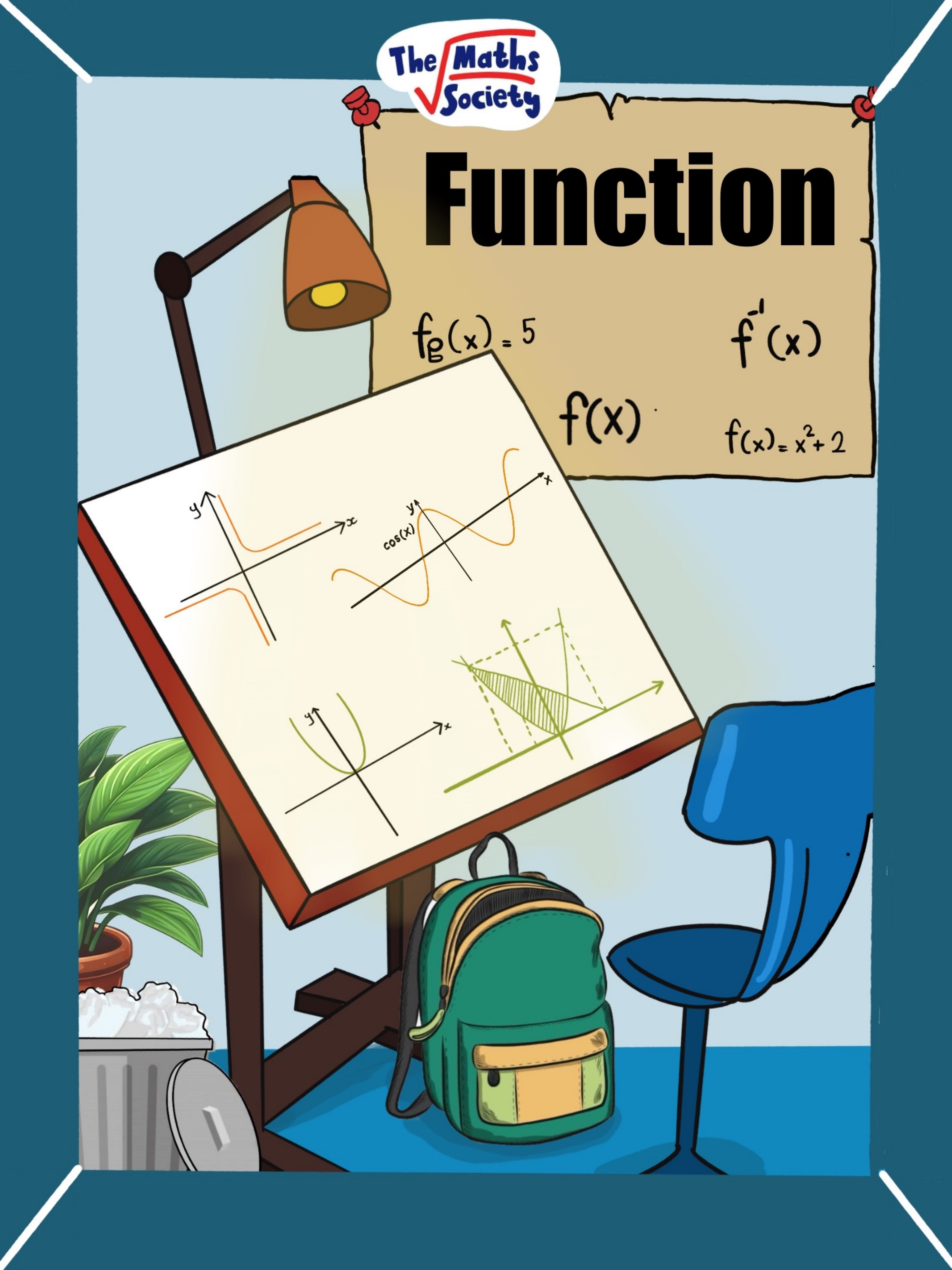
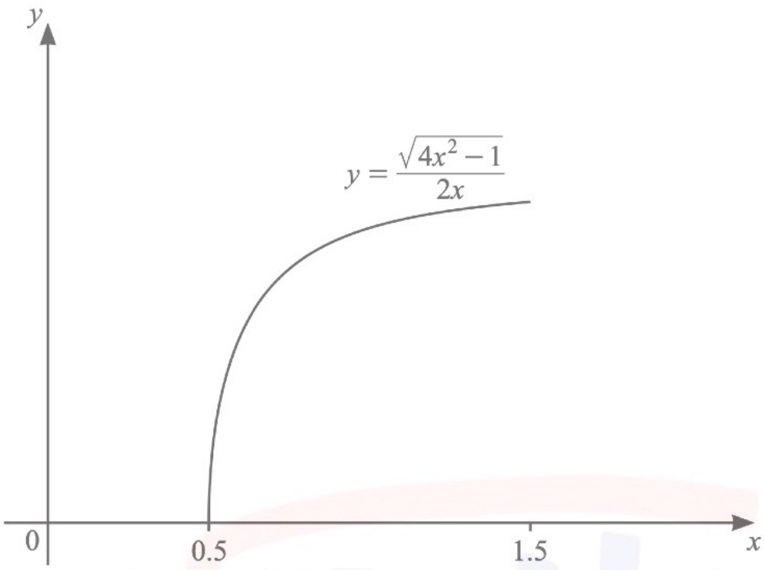

The Maths Society
Function
fg(x) = 5
f⁻¹(x)
f(x)
f(x) = x² + 2
cos(x)
y
x

1. The function f is defined by $f(x) = \frac{\sqrt{4x^2-1}}{2x}$ for $0.5 \le x \le 1.5$.
   The diagram shows a sketch of $y = f(x)$.



(a) (i) It is given that f^{-1} exists. Find the domain and range of f^{-1}.

[3]

$f(x)$: $\quad f^{-1}(x)$:

Domain $\Longleftrightarrow$ Range

$0.5 \le x \le 1.5$

Range $\Longleftrightarrow$ Domain

$$f(1.5) = \frac{\sqrt{4(1.5)^2-1}}{2(1.5)} = \frac{2\sqrt{2}}{3}$$

$\therefore$ For $f^{-1}(x)$ Domain: $0 \le x \le \frac{2\sqrt{2}}{3}$

Range: $0.5 \le f^{-1}(x) \le 1.5$

(ii) Find an expression for $f^{-1}(x)$.

[3]

Let $x = \frac{\sqrt{4y^2-1}}{2y}$

$2xy = \sqrt{4y^2-1}$

$4x^2y^2 = 4y^2-1$

$4y^2 - 4x^2y^2 = 1$

$4y^2(1-x^2) = 1$

$y = \frac{1}{2\sqrt{1-x^2}}$

$\therefore f^{-1}(x) = \frac{1}{2\sqrt{1-x^2}}$ for $0 \le x \le \frac{2\sqrt{2}}{3}$

(b) The function g is defined by $g(x) = e^{x^2}$ for all real x. Show that $gf(x) = e^{(1-\frac{a}{bx^2})}$, where a and b are integers.

[2]

$g(f(x)) = e^{\left[\left(\frac{\sqrt{4x^2-1}}{2x}\right)^2\right]}$

$= e^{\left[\frac{4x^2-1}{4x^2}\right]}$

$= e^{\left(1-\frac{1}{4x^2}\right)}$

2. It is given that $f(x) = 2\ln(3x - 4)$ for $x > a$.

(a) Write down the least possible value of a.

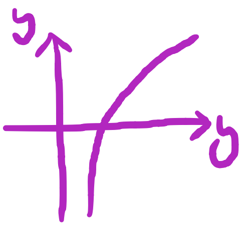
[1]

$3x - 4 > 0$

$x > \frac{4}{3}$ $\therefore a = \frac{4}{3}$

(b) Write down the range of f.

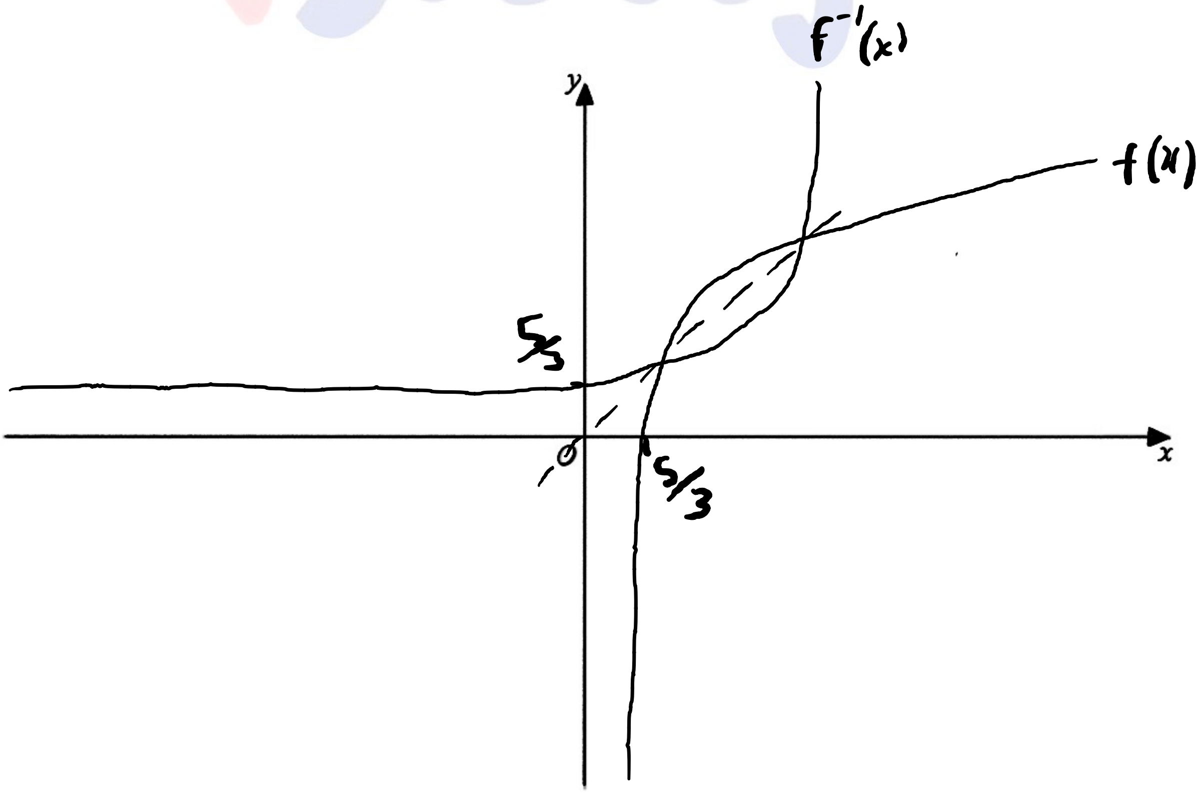
[1]



$y \in \mathbb{R}$

(c) It is that the equation $f(x) = f^{-1}(x)$ has two solutions. (You do not need to solve this equation). Using your answer to **part (a),** sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the axes below, starting the coordinates of the points where the graphs meet the axes.

[4]



It is given that $g(x) = 2x - 3$ for $x \geq 3$.

(d) (i) Find an expression for $g(g(x))$.

[1]

$$g(g(x)) = 2(2x-3)-3$$

$$= 4x - 9 \quad \text{for } x \geq 3$$

(ii) Hence solve the equation $fg(g(x)) = 4$ giving your answer in exact form.

[3]

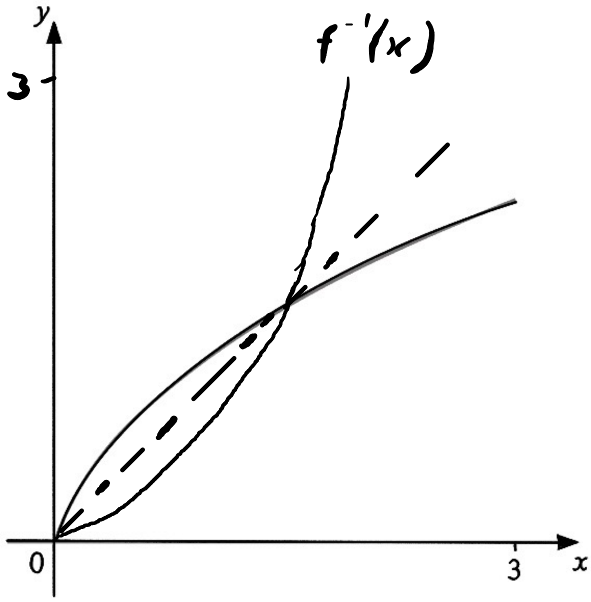
$$f(g(g(x))) \Rightarrow 2\ln[3(4x-9)-4] = 4$$

$$\ln[12x-31] = 2$$

$$12x - 31 = e^2$$

$$x = \frac{e^2+31}{12}$$

3.(a)



The diagram shows the graph of $y = f(x)$ where f is defined by $f(x) = \frac{3x}{\sqrt{5x+1}}$ for $0 \leq x \leq 3$.

(i) Given that f is a one-one function, find the domain and range of f^{-1}.

[3]

$f(x)$: $\quad f^{-1}(x)$:

Domain $\Leftrightarrow$ Range

$0 \leq x \leq 3$

Range $\Leftrightarrow$ Domain

$f(3) = \frac{3(3)}{\sqrt{5(3)+1}} = \frac{9}{4}$

$\therefore$ For $f^{-1}(x)$ Domain: $0 \leq x \leq \frac{9}{4}$

Range: $0 \leq f^{-1}(x) \leq 3$

(ii) Solve the equation $f(x) = x$.

[2]

$$\frac{3x}{\sqrt{5x+1}} = x$$

$$\sqrt{5x+1} = 3$$

$$5x = 8$$

$$x = \frac{8}{5}$$

(iii) On the diagram above, sketch the graph of $y = f^{-1}(x)$.

[2]

(b) The function g and h are defended by

$$g(x) = \sqrt[3]{8x^3 + 3} \quad \text{for } x \geq 1,$$
$$h(x) = e^{4x} \quad \text{for } x \geq k.$$

(i) Find an expression for $g^{-1}(x)$.

[2]

Let $x = \sqrt[3]{8y^3 + 3}$

$x^3 = 8y^3 + 3$

$y = \frac{\sqrt[3]{x^3 - 3}}{2}$

$\therefore g^{-1}(x) = \frac{\sqrt[3]{x^3 - 3}}{2}$

(ii) State the least value of the constant k such that $gh(x)$ can be formed.

[1]

$k = 1$

(iii) Find and simplify an expression for $gh(x)$.

[1]

$g(h(x)) = \sqrt[3]{8(e^{4x})^3 + 3}$

$= \sqrt[3]{8e^{12x} + 3}$

4.(a) The function f and g are defined by

$$f(x) = \sec x \qquad \text{for } \frac{\pi}{2} < x < \frac{3\pi}{2}$$

$$g(x) = 3(x^2 - 1) \qquad \text{for all real } x.$$

(i)Find the range of f.

[1]

$f(x) \leq -1$

(ii) Solve the equation $f^{-1}(x) = \frac{2\pi}{3}$.

[3]

$f^{-1}(x) = \sec^{-1}(x)$

$\sec^{-1}(x) = \frac{2\pi}{3}$

$x = \sec\left(\frac{2\pi}{3}\right)$

$= \frac{1}{\cos\left(\frac{2\pi}{3}\right)}$

$= -2$

(iii) Given that gf exists , state the domain of gf.

[1]

$\frac{\pi}{2} < x < \frac{3\pi}{2}$

(iv) Solve the equation $gf(x) = 1$.

[5]

$g(f(x)) = 3\sec^2 x - 3 = 1$

$\Rightarrow 3\sec^2 x = 4$

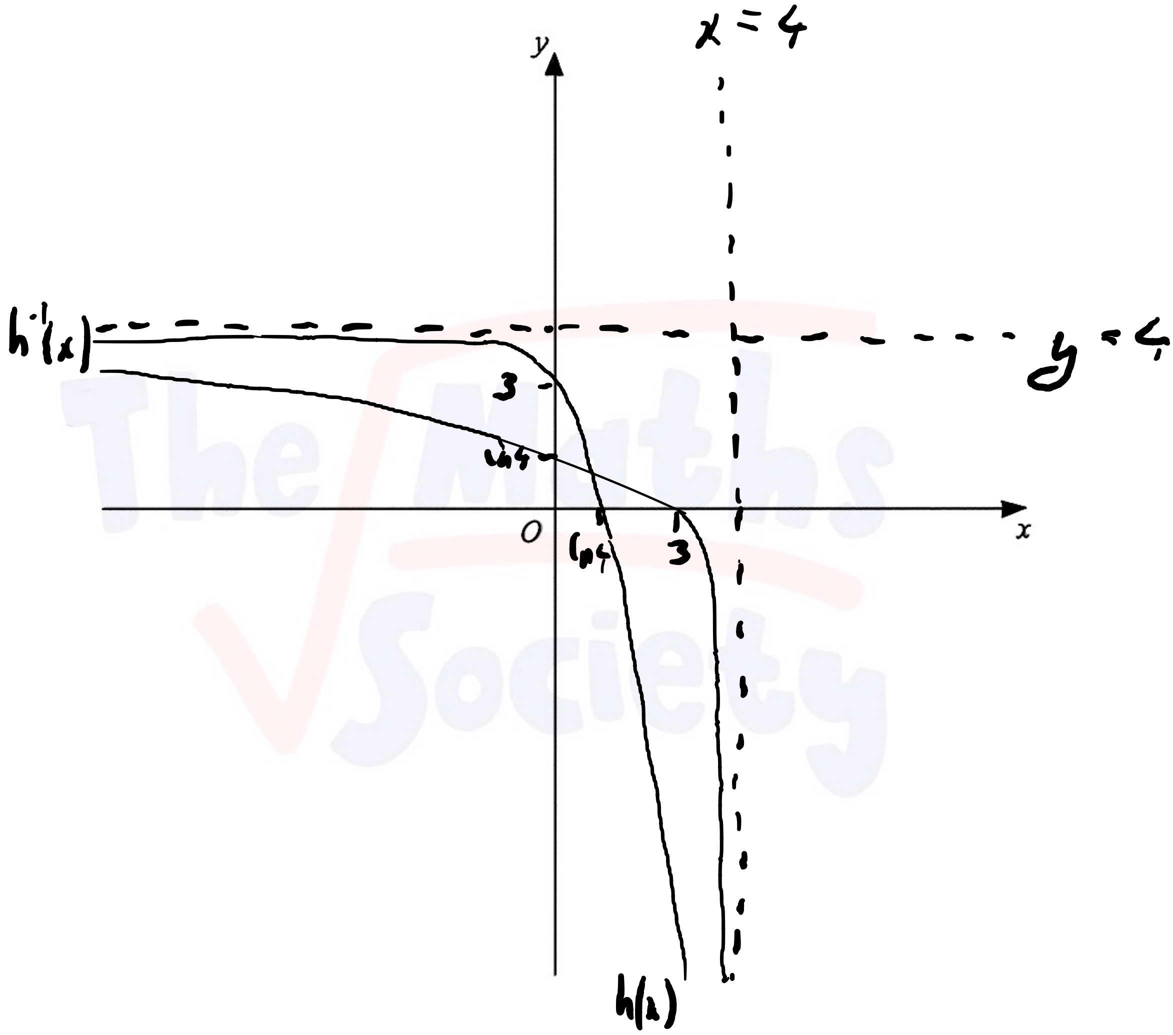
$\cos x = \frac{\sqrt{3}}{2}$

$x = \pm\frac{\pi}{6}$ but $\frac{\pi}{2} < x < \frac{3\pi}{2}$

$\therefore x = \frac{5\pi}{6}, \frac{7\pi}{6}$

(b)The function h is defined by $h(x) = \ln(4 - x)$ for $x < 4$. Sketch the graph of $y = h(x)$ and hence sketch the graph of $y = h^{-1}(x)$ .Show the position of any asymptotes and any points of intersection with the coordinate axes.

[4]



5. A function $f(x)$ is such that $f(x) = \ln(2x + 3) + \ln 4$, for $x > a$, where a is a constant.

(a) Write down the least possible value of a.

[1]

$$2x + 3 > 0$$

$$x > -\frac{3}{2} \quad \therefore a = -\frac{3}{2}$$

(b) Using your value of a, write down the range of f.

[1]

$$f(x) \in \mathbb{R}$$

(c) Using your value of a, find $f^{-1}(x)$, stating its range.

[4]

$$\text{Let } x = \ln(2y + 3) + \ln 4$$

$$= \ln(8y + 12)$$

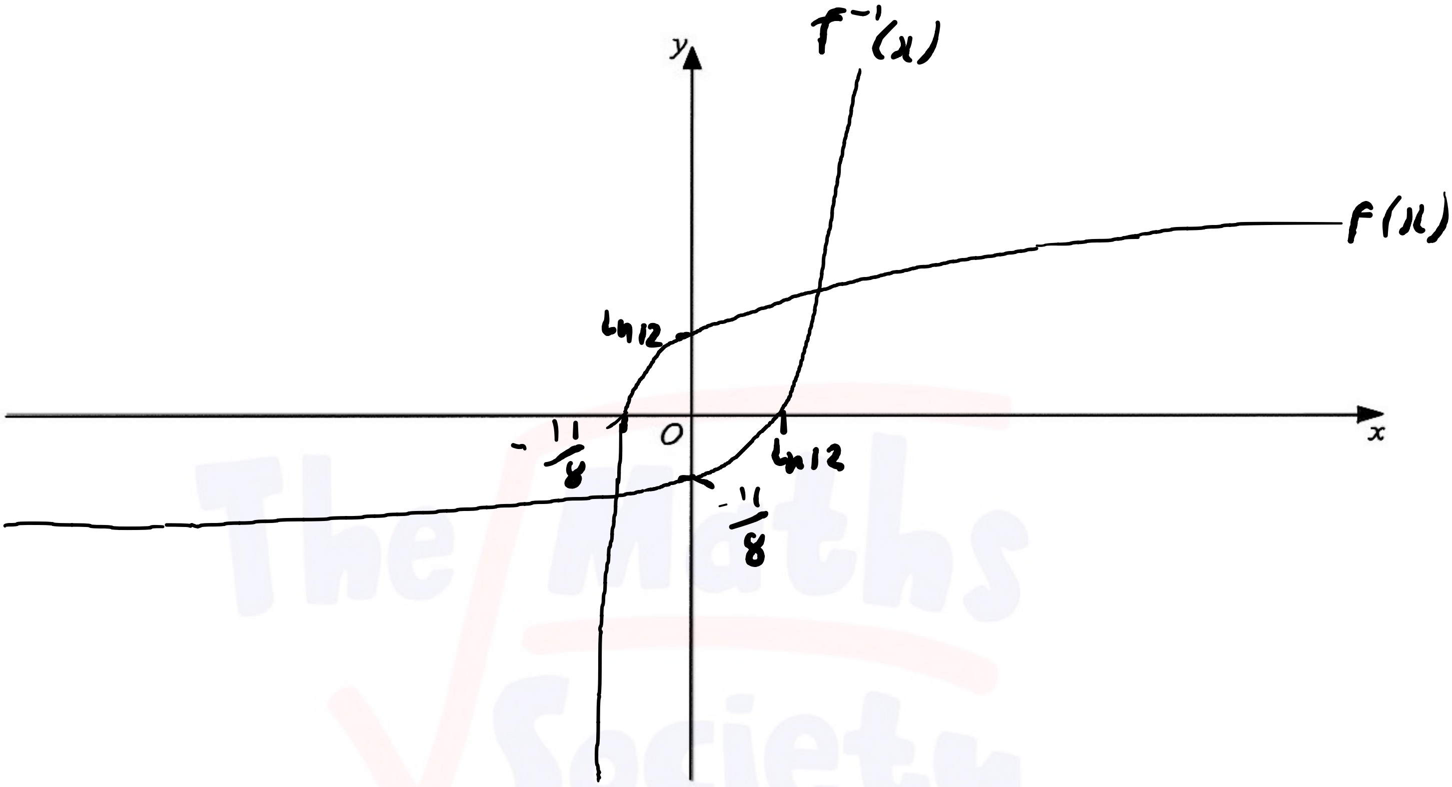
$$e^x = 8y + 12$$

$$y = \frac{e^x}{8} - \frac{3}{2}$$

$$\therefore f^{-1}(x) = \frac{e^x}{8} - \frac{3}{2} \ ; \qquad f^{-1}(x) > -\frac{3}{2}$$

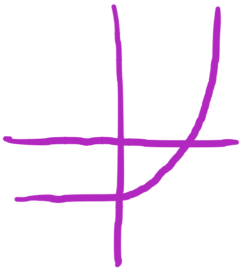
(d) On the axes below, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, stating the exact intercepts of each graph with the coordinate axes. Label each of your graphs.

[4]



6 . A function $f(x)$ is such that $f(x) = e^{3x} - 4$, for $x \in \mathbb{R}$.

(a) Find the range of f.



$f(x) > -4$

[1]

(b) Find an expression for $f^{-1}(x)$.

[2]

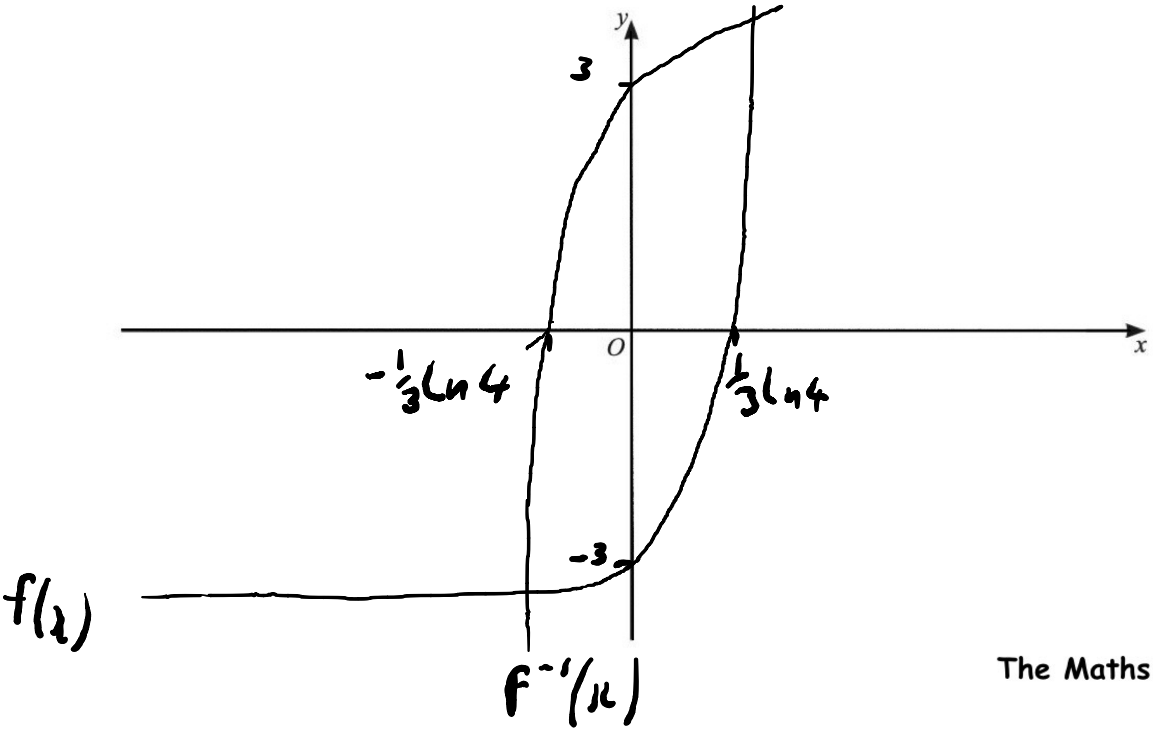
Let $x = e^{3y} - 4$

$3y = \ln(x+4)$

$y = \frac{1}{3}\ln(x+4)$

(c) On the axes, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ stating the exact values of the intercepts with the coordinate axes.

[4]



7. The functions $f(x)$ and $g(x)$ are defined as follows for $x > -\frac{1}{3}$ by

$$f(x) = x^2 + 1,$$
$$g(x) = \ln(3x + 2).$$

(a) Find $fg(x)$.

[1]

$$f(g(x)) = [\ln(3x+2)]^2 + 1$$

(b) Solve the equation $fg(x) = 5$ giving your answer in exact form.

[3]

$$[\ln(3x+2)]^2 + 1 = 5$$
$$\ln(3x+2) = 2$$
$$3x+2 = e^2$$
$$x = \frac{e^2-2}{3}$$

(c) Solve the equation $gg(x) = 1$.

[6]

$$g(g(x)) = \ln\left(3\left[\ln(3x+2)\right]+2\right) = 1$$

$$\Rightarrow \quad 3\ln(3x+2)+2 = e$$

$$\ln(3x+2) = \frac{e-2}{3}$$

$$3x+2 = e^{\left(\frac{e-2}{3}\right)}$$

$$x = \frac{1}{3}\left(e^{\left(\frac{e-2}{3}\right)}-2\right)$$